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RESEARCH LABORATORY OF ELECTRONICS  
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IN THE PRESENCE OF PLASMA RESONANCE

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Abstract

In a high frequency discharge, plasma resonance maximizes the electric field, thus producing a high ionization rate in the regions near resonance. The effect on the distribution of electrons and of ionization in a parallel plane discharge is calculated and compared with the observation that the light from such a discharge often is a minimum at the center.

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# ELECTRON DENSITY DISTRIBUTION IN A HIGH FREQUENCY DISCHARGE IN THE PRESENCE OF PLASMA RESONANCE

## I. Introduction

It was pointed out by Schumann (1) that when a high frequency field is applied to an ionized gas of high electron concentration, the field tends to concentrate in those parts of the discharge where the plasma is near resonance (2). The admittance of a high frequency discharge under such conditions was discussed in a recent paper (3). The appearance of a microwave discharge at high current is shown in Fig. 1 in which the discharge is taking place between parallel plates at the right and left of the bright glow, and it is noted that the light intensity is greatest near the electrodes. It may be inferred that the field is greatest here, and this corresponds, according to Schumann, to regions where the plasma is in resonance and is shielding the central part of the discharge. It is the purpose of this report to calculate the electron density throughout the discharge under such conditions, and to show that the ionization has sharp maxima in the resonance regions.

## II. The Ionization Function

For a given applied field of radian frequency  $\omega$ , plasma resonance occurs at an electron density (2)

$$n = \frac{m\omega^2\epsilon_0}{e^2} \quad (1)$$

where  $e$  and  $m$  are the charge and mass of an electron, and  $\epsilon_0$  is the electric permittivity of free space. It is convenient to measure electron concentration in terms of the variable

$$r = \frac{ne^2}{m\omega^2\epsilon_0} \quad (2)$$

which is unity when the plasma is in resonance. A damping constant for the plasma is given by

$$\beta = \frac{\nu_c}{\omega} \quad (3)$$

where  $\nu_c$  is the average frequency of collision of an electron with the molecules of the gas. The formula for the total current density, including both electron current and displacement current, is given by Eqs. 5 and 9 of reference 3. In terms of the parameters defined above it is

$$J = \left[ \frac{ne^2}{m(\nu_c + j\omega)} + j\omega\epsilon_0 \right] E = \left[ \frac{r(\beta - j)}{(\beta^2 + 1)} + j \right] \omega\epsilon_0 E \quad (4)$$

from which the magnitude of the resistivity is

$$\left| \frac{E}{J} \right| = \left( \frac{1}{\omega \epsilon_0} \right) (\beta^2 + 1)^{1/2} [(r-1)^2 + \beta^2]^{-1/2}. \quad (5)$$

This quantity has a maximum at resonance where  $r = 1$ .

The ionization frequency per electron  $\nu_i$  depends on the applied field, and over a limited range about a central value  $E_0$  we shall assume, following Herlin and Brown (4), that it varies as a power  $2a$  of the field

$$\frac{\nu_i}{\nu_0} = \left| \frac{E}{E_0} \right|^{2a}. \quad (6)$$

With the value of  $E$  from Eq. 5 this is

$$\nu_i = \nu_0 \left| \frac{J}{\omega \epsilon_0 E_0} \right|^{2a} (\beta^2 + 1)^a [(r-1)^2 + \beta^2]^{-a}. \quad (7)$$

The ionization rate is thus a function of the electron density with a maximum at  $r = 1$ .

### III. The Diffusion Equation

The distribution of electrons in the discharge is determined by the production rate  $n\nu_i$  in the gas and diffusion to the walls, and it will be assumed that the electron density is such that ambipolar diffusion exists. Limiting ourselves to the case of extended plane parallel plates perpendicular to the  $z$ -axis, the diffusion equation is (ref. 3)

$$\frac{d^2 n}{dz^2} + \frac{n\nu_i}{D_a} = 0 \quad (8)$$

where  $D_a$  is the ambipolar diffusion coefficient. It is convenient to normalize distance by the variable  $u = 2z/\delta$ , where  $z$  is measured from the midplane and  $\delta$  is the plate separation.

After substituting from Eq. 7 we then express Eq. 8 in terms of  $r$  and  $u$

$$\frac{d^2 r}{du^2} + \frac{k^2 r}{(r^2 - 2r + 1 + \beta^2)^a} = 0 \quad (9)$$

where

$$k^2 = \left[ \frac{\delta^2 \nu_0 (1 + \beta^2)^a}{4D_a} \right] \left| \frac{J}{\omega \epsilon_0 E_0} \right|^{2a} \quad (10)$$

is a constant. It is determined as a characteristic value for Eq. 9 with the boundary conditions

$$\begin{aligned} r &= r_0, & \frac{dr}{du} &= 0, & \text{at } u &= 0 \\ r &= 0, & & & \text{at } u &= \pm 1. \end{aligned} \quad (11)$$

Equation 9 integrates once to give

$$\left(\frac{dr}{du}\right)^2 = k^2 \int_r^{r_0} \frac{2r dr}{(r^2 - 2r + 1 + \beta^2)^\alpha} = k^2 L^2. \quad (12)$$

The function  $L$  introduced here is proportional to the electron density gradient and therefore also to the electron diffusion current. A second integration gives the electron density as an implicit function of position

$$ku = \int_r^{r_0} \frac{dr}{L}. \quad (13)$$

The characteristic value of  $k$  is then given by setting the lower limit  $r = 0$  with  $u = 1$ .

#### IV. Integration

The integration in Eq. 12 is readily performed for any integral or half-integral value of  $\alpha$  but the integration in Eq. 13 causes trouble. We shall carry it out with four different approximations applicable to different discharges or to different parts of the same discharge. From these the general form of the solution may be readily visualized. The four approximations will be

A.

$$2r \ll r^2 + 1 + \beta^2.$$

By neglecting the only negative term in the denominator of Eq. 12 all resonance effects on the electron distribution are eliminated. This approximation is always good at high pressures  $\beta^2 \gg 1$  and also at any pressure for electron concentrations not near resonance. The next two approximations are subgroups of A.

B.

$$r^2 - 2r \ll 1 + \beta^2.$$

This amounts to neglecting the alternating electron current relative to the displacement current or to assuming a uniform field. It is good only for electron concentrations well below resonance or for sufficiently high pressures.

C.

$$\beta^2 + 1 \ll r^2.$$

This approximation neglects the displacement current. It is good only at electron concentrations above resonance and therefore never good near the walls.

D.

$$\beta \ll 1.$$

This neglects the resistive current and is the low pressure limit. The effects of resonance dominate this approximation.

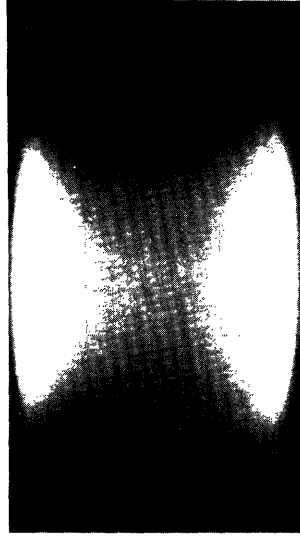


Fig. 1  
Appearance of the high frequency discharge.

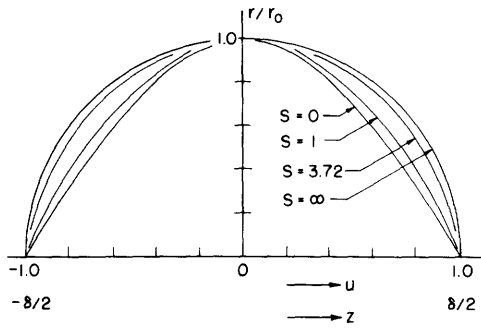


Fig. 2

Variation of electron density  $r$  across a discharge for which  $\alpha = 2$  in the limit of high pressures for various ratios  $s = r_0(\beta^2 + 1)^{-1/2}$ .

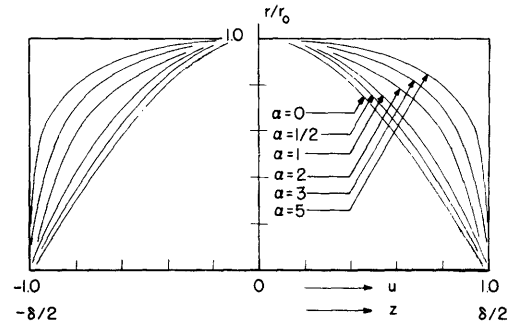


Fig. 3

Variation of electron density  $r$  across a discharge. Limiting form at very high electron concentrations  $s \gg 1$  for various values of  $\alpha$ .

A. Neglecting the term  $2r$  in the denominator of Eq. 12 the integration yields, for  $\alpha \neq 1$

$$L^2 = \frac{[(r^2 + 1 + \beta^2)^{1-\alpha} - (r_0^2 + 1 + \beta^2)^{1-\alpha}]}{(\alpha-1)}. \quad (14)$$

Substitution in Eq. 13 then leads to incomplete elliptic integrals (5) for all integral values of  $\alpha$ . Substituting

$$\chi = r_0(r_0^2 + 1 + \beta^2)^{-1/2} \quad \text{and} \quad \cos \phi = \frac{r}{r_0} \quad (15)$$

we find for  $\alpha = 2$



$$ku = (r_o^2 + 1 + \beta^2) E(\kappa, \phi) \quad (16)$$

and for  $\alpha = 3$

$$ku = (r_o^2 + 1 + \beta^2)^{3/2} \left[ 2E\left(\frac{\kappa}{\sqrt{2}}, \phi\right) - F\left(\frac{\kappa}{\sqrt{2}}, \phi\right) \right]. \quad (17)$$

Curves of  $r/r_o$  vs.  $u$  for  $\alpha = 2$  and various values of  $r_o(\beta^2 + 1)^{-1/2} = s$  are shown in Fig. 2. The two limiting curves in this case ( $\alpha = 2$ ) are a cosine and a circle, the effect of increasing electron concentration being always to increase the concentration near the walls relative to that in the center.

B. Neglecting  $r^2 - 2r$  in the denominator of Eq. 12 leads to elementary integrations

$$L^2 = \frac{(r_o^2 - r^2)}{(1 + \beta^2)^\alpha} \quad (18)$$

$$r = r_o \cos \left[ \frac{ku}{(1 + \beta^2)^{\alpha/2}} \right] = r_o \cos \frac{\pi u}{2} \quad (19)$$

$$k = (1 + \beta^2)^{\alpha/2} \frac{\pi}{2}. \quad (20)$$

This is the solution of the simple diffusion equation and substitution of this last expression in Eqs. 10 and 7 leads to the usual equilibrium condition

$$v_i = \frac{\pi^2 D_a}{\delta^2}. \quad (21)$$

The field and ionization rate per electron are, in this case, uniform throughout the discharge.

C. The neglect of all but  $r^2$  in the denominator of Eq. 12 leads to the integral

$$ku = (\alpha - 1)^{1/2} r_o^{\alpha-1} \int_r^{r_o} r^{\alpha-1} (r_o^{2\alpha-2} - r^{2\alpha-2})^{-1/2} dr. \quad (22)$$

This integration has been performed for a number of values of  $\alpha$  and the results are shown in Fig. 3. For  $\alpha = 0$  the curve is a cosine, for  $\alpha = 2$  it is a circle, and as  $\alpha \rightarrow \infty$  it approaches a rectangular box.

D. The integrand of Eq. 12 has singularities at  $r = 1 \pm j\beta$  but the approximations made in A, B and C put these singularities back on the imaginary axis so that the path of integration did not go between them. At low pressures these singularities lie close to, and on either side of, the real axis, and the integration in Eq. 12 must pass between them if  $r_o > 1$ . The major part of the integral then comes, for small  $\beta$ , from the saddle-point region near  $r = 1$ .  $dr/du$  changes rapidly in this region and there is a bend in the plot of  $r$  vs.  $u$ . In order to obtain the value  $L_o$  of  $L$  at the boundary we may set  $r = 1$



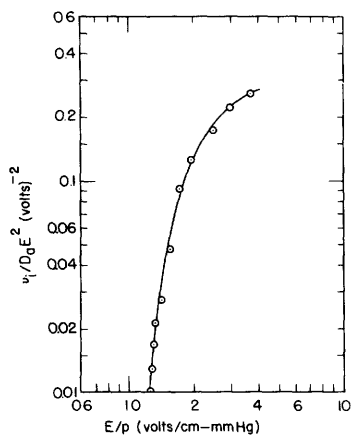


Fig. 5  
Variation of  $\nu_i/D_a E^2$  with  $E/p$   
in helium for large  $\beta$ .

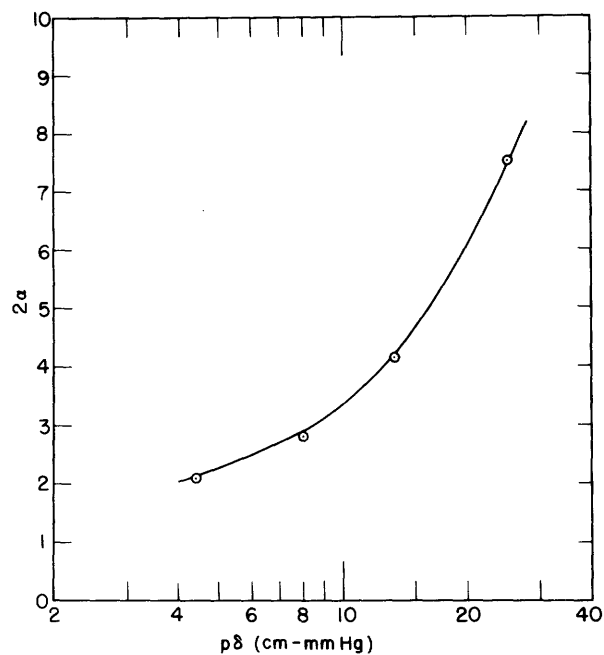


Fig. 6  
Variation of  $\alpha$  with  $p\delta$  in helium for large  $\beta$ .

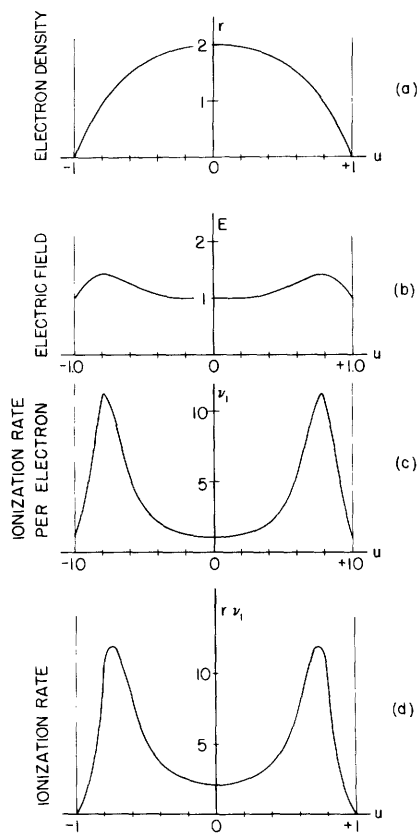


Fig. 7  
Variation of electron density, electric field,  
and ionization rate across the discharge.

These relations are particularly simple for  $\alpha = 2$ . Then  $L_0^2 = \pi/\beta^3$ ,  $k = r_0(r_0^2 - 1)^{1/2} + 1/L_0$ ,  $1/u_1 = 1 + 1/L_0 r_0(r_0^2 - 1)^{1/2}$ . Curves for this case are shown in Fig. 4. These curves have sharp bends at the resonance points because of the approximations made. The lowest curve does not reach resonance and is a cosine.

## V. Determination of $\alpha$

One method of determining  $\alpha$  experimentally will be described to show that at a given frequency there is a relationship between  $\alpha$  and  $\beta$ . The values of  $\alpha$  can be obtained from measurements of the operating field  $E$  at various pressures  $p$  when the electron concentration is kept below resonance but not so small that ambipolar diffusion does not prevail. Under these conditions  $E$  is constant at a given pressure independent of both discharge current and position. Therefore  $v_i/D_a$  will also be constant; equal, in fact, to  $\pi^2/\delta^2$  as is given by Eq. 21. Data are thus obtained for making a plot of  $v_i/D_a E^2$  vs.  $E/p$ , which are the proper variables for the problem as Herlin and Brown(4) have shown. The slope of this curve on logarithmic paper is  $2\alpha - 2$ , from Eq. 6, and thus  $\alpha$  is determined at various electric fields and pressures.

The procedure outlined above is essentially similar to that of Herlin and Brown, who treated the breakdown condition instead of the steady state discharge studied here. They point out further that there is a curve of  $v_i/D_a E^2$  vs.  $E/p$  for each value of  $p\lambda$ , where  $\lambda$  is the free space wavelength of the exciting field, and  $\beta = 0.013 p\lambda$  (cm-mm Hg) for helium.

Data for the limiting case of large  $\beta$  have been obtained for helium and plotted in Fig. 5. The corresponding values of  $\alpha$  vs.  $p\delta$  obtained from the slope of this curve are plotted in Fig. 6.

## VI. Distribution of Ionization

The electric field and ionization distribution will be computed in detail for a particular discharge in helium. For a value of  $p\delta$  of 23 we find from Fig. 6 that  $\alpha = 7/2$ . Also if  $p\lambda = 77$  we find that  $\beta = 1$ . We further choose the electron density at the center of the discharge to be such that  $n\lambda^2 = 2.2 \times 10^{13}/\text{cm}$  so that  $r_0 = 2$ . The curve of electron density vs. position for these values is shown in Fig. 7a. The variation of electric field across the discharge is computed from Eq. 5 and shown in Fig. 7b. Equation 6 can then be used to compute Fig. 7c which shows the variation of the ionization rate per electron at various positions across the discharge. The ionization rate will be the product of the ionization rate per electron by the electron concentration and is thus the product of the curve of Fig. 7c with that of Fig. 7a. This is shown in Fig. 7d. Assuming that the excitation at each point is proportional to the ionization at that point this should also correspond to the light intensity emitted by various portions of the discharge.

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(Our  $\alpha$  is  $\beta/2$  in their notation.)
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